THE GRAPH SPECTRUM OF BARYCENTRIC REFINEMENTS

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ABSTRACT. Given a finite simple graph G, let G_1 be its barycentric refinement: it is the graph in which the vertices are the complete subgraphs of G and in which two such subgraphs are connected, if one is contained into the other. If $\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Laplacian of G, define the spectral function $F(x) = \lambda_{[nx]}$ on the interval $[0,1]$, where $[r]$ is the floor function giving the largest integer smaller or equal than r. The graph G_1 is known to be homotopic to G with Euler characteristic $\chi(G_1) = \chi(G)$ and $\dim(G_1) \geq \dim(G)$. Let G_m be the sequence of barycentric refinements of $G = G_0$. We prove that for any finite simple graph G, the spectral functions F_{G_m} of successive refinements converge for $m \to \infty$ uniformly on compact subsets of $(0,1)$ and exponentially fast to a universal limiting eigenvalue distribution function F_d which only depends on the clique number respectively the dimension d of the largest complete subgraph of G and not on the starting graph G. In the case $d = 1$, where we deal with graphs without triangles, the limiting distribution is the smooth function $F(x) = 4\sin^2(\pi x/2)$. This is related to the Julia set of the quadratic map $T(z) = 4z - z^2$ which has the one dimensional Julia set $[0,4]$ and F satisfies $T(F(k/n)) = F(2k/n)$ as the Laplacians satisfy such a renormalization recursion. The spectral density in the $d = 1$ case is then the arc-sin distribution which is the equilibrium measure on the Julia set. In higher dimensions, where the limiting function F still remains unidentified, F' appears to have a discrete or singular component. We don't know whether there is an analogue renormalization in $d \geq 2$. The limiting distribution has relations with the limiting vertex degree distribution and so in 2 dimensions with the graph curvature distribution of the refinements G_m.

1. INTRODUCTION

The spectral theory of graphs [6, 43, 1, 7, 42, 47] parallels to a great deal the spectral theory of compact Riemannian manifolds [4, 44, 3]. There are areas with a good match like inverse spectral topics, heat kernel related topics like Hodge theory where the dimension of the space of harmonic solutions $Lf = 0$ for example is the number of connected components and more generally, harmonic k-forms are related to the k'th cohomology groups in a rather explicit

way as applying the heat flow to k forms converges to harmonic forms. This can be used for example to construct the Hurewicz homomorphism from $\pi_k(G)$ to the cohomology groups $H^k(G)$ by applying the heat flow e^{-iL_k} on a k-form f supported initially on a k-sphere representing an element in the homotopy group $\pi_k(G)$. It leads to a harmonic k-form representing a homology class. Discrete McKean-Singer [26] $\mathrm{str}(e^{-tL}) = \chi(G)$ illustrates further, how the graph theory parallels the continuum [40]: the super trace of the heat kernel for $t = 0$ is the super trace of 1 which is the definition of the Euler characteristic, while in the case $t \to \infty$, it counts the alternating sum of the Betti numbers, which by Euler-Poincaré is the same. The cohomology of discretizations of manifolds has since the beginning of the development of algebraic topology been used to compute the cohomology of the manifold. For the convergence of the spectrum of p-form Laplacians of a compact connected Riemannian manifold M of dimension $d \geq 2$, Mantuano's theorem [39] tells that for any ϵ discretization G given by a graph, the spectra of the Hodge Laplacians L_p are related by $c\lambda_{k,p}(G) \leq \lambda_{k,p}(M) \leq C\lambda_{k,p}(G)$, where the constants c, C only depend on dimension d, the maximal curvature and radius of injectivity of M. Also for eigenfunctions, the number of nodal regions of the k eigenfunction f_k is bound by a theorem of Fiedler [11] by k, paralleling the Courant nodal theorem. The ground state energy $\lambda_1 > 0$ is estimated in the same way from below by the Cheeger constant. The nodal regions on geometric graphs correspond to Chladni figures on compact Riemannian manifolds. Also in topology like for Brouwer-Lefshetz [27] or Jordan-Brouwer theory [34], the notions translate nicely to the discrete and barycentric refinements help. There are also places, where things are different: the Weyl law relating the growth of the eigenvalues with volume has no direct discrete analogue because the spectrum of a graph is a finite set. Similarly, the Minashisundaram-Plejel zeta function $\sum_{\lambda>0} \lambda^{-s}$ of a manifold needs analytic continuation in the continuum, while in the graph case, the zeta function is an entire function. Already in the case of a circle, where the Dirac version of that zeta function is the Riemann zeta function, the zeros of the discrete analog on circular graphs can be analyzed well, unlike in the continuum, where it it is the Riemann hypothesis. The convergence of the roots [36] of the graph zeta functions is a limit when looking at barycentric subdivisions of circular graphs, which is also important for Jacobi matrices generalizing graph Laplacians [41, 14] which we studied in [22] in an ergodic setup, where the limiting operators are almost periodic Jacobi matrices over the von Neumann-Kakutani system. The later is the unique fixed point of the 2:1 integral extension operation in ergodic theory. The hull of the operators is the compact Abelian group of dyadic integers, where space translation is addition by 1 and where the renormalization step is the shift. We look now at higher dimensional analogue questions to these one dimensional Jacobi spectral problems. There are other analogies between Schrödinger operators and geometric Laplacians: while isospectral deformations are possible in the former case, geometric Laplacians are harder
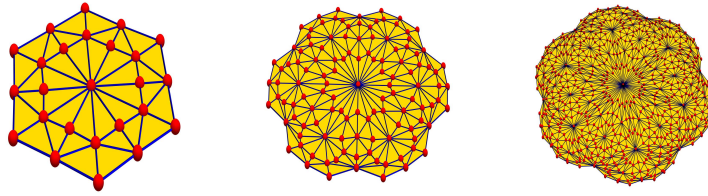
FIGURE 1. Barycentric refinements of the triangle $G = K_3$. The number of vertices of the ball G_m grows exponentially like $O(6^m)$ and is exactly known.

to deform as the isospectral set of geometries is discrete in general. Having worked with isospectral deformations of Jacobi matrices in [20, 19] we looked in [21] at higher dimensions and noted that writing $L = D^2$ can enable isospectral deformation and break the spectral rigidity. Much later, we realized that in the Riemannian as well as in the graph case, one can deform the exterior derivative d [30, 29] in an isospectral way. The Riemannian case is a bit more technical in that story as the deformed exterior derivatives d_t are pseudo differential equations in the continuum but they satisfy $d_t^2 = 0$ so that they deform cohomology. The deformation does not affect the Laplacian L, but in the complex, the nonlinear deformation becomes asymptotic to the wave evolution. Interestingly, both in the discrete as well as in the continuum, space expands, with an inflationary start. This is not artificially placed into the model, but is a basic property if one lets the Dirac operator move freely in its symmetry group. Graphs can serve as a laboratory to test concepts related to physics. It is reasonable for example to see the evolutions of the various differential forms under the wave dynamics as manifestations of different forces. It is a caricature for physics, where nothing has put into the system except for geometry, which is a Riemannian manifold in the continuum and a graph G in the discrete. When answering the question which graph to take, symmetry again could give a hint: take a scale invariant graph. While impossible to achieve in a finite discrete structure, taking a graph which is multiple way refined in a barycentric way comes close. Not only are parts of the graph homeomorphic to the entire graph, but also the nerve graphs of open covers are homeomorphic to the entire graph. This motivates to look at refinements and its spectral properties.

In [35], we looked at a product of graphs which satisfies the same Künneth formula for cohomology as in the continuum. This product of G and H is defined as follows: take two disjoint unions of G and H. Pick a complete subgraph x of G and a complete subgraph y of H. Let xy be the complete subgraph generated by $x \cup y$. These points xy form the vertices of $G \times H$. Two such vertices xy and uv are connected if one is contained in the other.
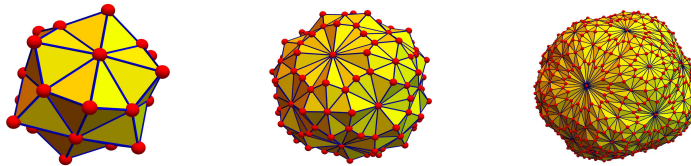
FIGURE 2.   Barycentric refinements of the octahedron.

A special case is if $H = K_1$ in which case $G_1 = G \times K_1$ is the barycentric
refinement of G: its vertices are the complete subgraphs x of G and two
complete subgraphs x, y are connected if one is contained in an other. The
graph G_1 is homotopic to G. It especially has the same Euler characteristic.
The dimension of G_1 is bounded below by the dimension of G. If G is
geometric, then G_1 is even homeomorphic to G and the dimensions of G and
G_1 are the same. For any d-dimensional geometric G, the graphs G_m are
all $(d + 1)$-colorable, the color being the dimension of the original simplices
which make up now the vertices. The automorphism group of G also acts
on G_1 so that fixed points of graph automorphisms [27] can be realized as
vertices in G_1. Barycentric refinements again stress the point of view taken
by discrete Morse theory [12, 13] that complete subgraphs in a graph can
be treated as "points".
For practical triangulations, barycentric refinements are rarely used. The
reason is the huge difference between the possible vertex degrees, as each
refinement doubles the maximal degree. But the appearance of large ver-
tex degrees has also benefits: for any m, the G_m graphs are Eulerian with
Eulerian spheres and have the property that a natural geodesic flow can be
defined on them in any dimension. This is essential, as spheres and lines are
important in any geometry and that for defining lines, we need an Eulerian
property as with an odd degree vertex, the continuation of a "straight line"
is ambiguous: we have difficulty for example to continue a line through a
vertex of an icosahedron as we will have to chose what direction to continue.
For graphs in which all spheres and all spheres in spheres are Eulerian, there
is a natural continuation. Graph products and especially barycentric refine-
ments have this property. The large and nonuniform degree of some density
distributed vertices has the advantage as it allows to emulate the rotational
symmetry in the discrete. For a d-dimensional graph, and considering G_m,
we can in dimension d start with a vertex degree of $O((d + 1)!^m)$ direc-
tions. Unlike in regular lattices or tessellations, the spheres appear more
rounded and the fact that the unit sphere $S(x)$ has asymptotically a sim-
ilar amount of directions than the number of vertices in G_m gives us the
property that the exponential map $\exp_x$ from the unit sphere to the graph,
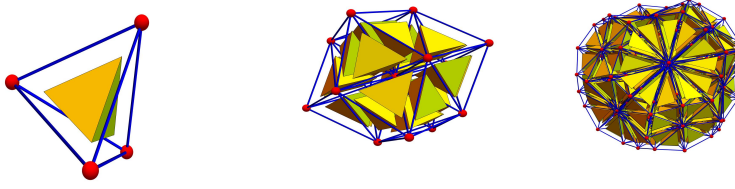covers most of the graph. This is probably the closest we can get to a

FIGURE 3. Barycentric refinements of the tetrahedron $G = K_4$. The number of vertices grows like $O(24^m)$. We know $v_4(G_m) = 24^m$ but we don't have an explicit formula yet for the number of vertices $v_0(G_m)$ if $G_0 = K_4$.

Hopf-Rynov property in the discrete without digressing into the quantum (as nature does) and look at the wave equation $u_{tt} = Lu$ for the Hodge Laplacian $L = (d+d^*)^2$, which because $L = D^2$ has the d'Alembert solution $u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(u)$, a superposition of wave group solutions $e^{\pm iDt} = \cos(Dt) \pm i\sin(Dt)$ of the Schroedinger equation $\psi' = \pm iD\psi$ with $\psi = u + iD^{-1}u'$ for the Dirac operator $D = d + d^*$ (see [28]) for which Hopf-Rynov is just linear algebra in the graph case as the unitary group is finite dimensional and for any two vertices x, y one can solve the problem to start a geodesic on e_x and reach e_y by choosing the correct velocity $u'(0)$ and time t.

Iterated barycentric subdivisions have long been used in topology and homology theory. The notion is mostly defined in geometric realizations used to refine a given triangulation of a simplex in Euclidean space. Note that we don't look at any embedding in an Euclidean space but look at Barycentric refinement in a completely combinatorial setup. There are also probabilistic connections [8, 15], as iterated barycentric subdivisions of a triangle define a random walk on $\mathrm{SL}(2, R)$ and are linked to concrete geometric properties like angles triangles. Such work could suggest that unlike in the one dimensional case, in higher dimensions, higher dimensional group actions and random walks could play a role when studying the spectrum. In graph theory, where no embedding into an Euclidean space is required, barycentric subdivisions are known in the context of flag complexes or Whitney complexes in particular. We are not aware of any work on the nature of the spectrum of such graphs.

The inductive dimension of a graph G is defined inductively as the average of the dimensions of all unit spheres, incremented by 1 [25, 23]. A graph G is a geometric d-dimensional graph, if every unit sphere is a $(d-1)$-sphere or a $(d-1)$-ball and the boundary set of vertices with a ball unit sphere form a $(d-1)$-dimensional geometric graph without boundary. Geometric

graphs play the role of manifolds with or without boundary. The definition of graph theoretical spheres and homotopy are both due to Evako. We have defined spheres independently in [33, 35] but realized in [34] that it is already in the work of Ivashchenko=Evako [17, 16, 9, 10]. The definition of these Evako spheres is based on Ivashchenko homotopy which is homotopy notion inspired by Whitehead [48] but defined for graphs. A simplification done in [5] makes this homotopy much more practical, like for Lusternik-Schnirelman theory for graphs [18], to define what a homeomorphism for graph is [32] or for graph colorings [31, 33], where constructing an Eulerian 3 sphere having a given 2 sphere embedded could explain why 4 colors are enough to color a planar graph.

We proved in [35] the inequality $\dim(G \times H) \geq \dim(G) + \dim(H)$, which holds for all finite simple graphs G, H. This implies in a special case $\dim(G_1) \geq \dim G$), still for all finite simple graphs G. In the geometric case, the dimension of G_1 is the same as the dimension of G. The barycentric refinement process $G \to G_1$ honors geometric graphs, graphs for which the unit spheres are spheres. Starting with a d-simplex $G_0 = K_{d+1}$, successive refinements G_m produce geometric graphs which are d-balls, for which the boundary is a $(d-1)$-sphere. In the case of a triangle $G = K_3$ for example, the graph G_1 is a wheel graph with 6 spikes. It has one interior point and 6 boundary point. Successive refinements produce larger and larger balls for which the boundaries are spheres.

If G is a geometric d-dimensional graph, then G_1 is homeomorphic to G in the sense of [32]: there is an open cover U_j of G_1 such that the nerve graph of G_1 is G. To see this, start with unit disks V_j centered at the vertices of G_1 which originally were vertices. Then add to V_j the d-simplices on the outside which contain $(d-1)$ simplex in V_j and a vertex simplex z. This produces U_j. Two different U_j intersect in a d-simplex, if and only the original vertices were connected. Homeomorphic graphs have the same topological properties like cohomology, fundamental groups or connectivities. We furthermore know that for any starting point G, the dimension of G_m converges to the dimension of the largest complete subgraph in G [35] and the reason is the same as for the spectral convergence studied here: the largest dimensional simplices dominate.

2. Counting

The refinements G_m grow exponentially fast already in the 1-dimensional case, where the number of edges doubles exactly in each step.

Lemma 1. *a) For $G = K_2$, $v_0(G_m)1 + 2^m$ and $v_1(G_m) = 2^m$.*
b) For $G = K_3$, the number of vertices $v_0(G_m) = 1 - 3(2^{m-1} + 2^m + 2^{m-1}3^{m-1})$, the number of edges is $v_1(G_m) = 3(-2^{m-1} + 2^m + 2^{m-1}3^m)$

and the number of triangles $v_2(G_m) = 6^m$.
c) For $G = K_d$, *we have* $v_d(G_m) = ((d+1)!)^m$ *and* $v_0(G_m) \geq ((d+1)!)^{m-1}$.

Proof. a) We have $(v_0(0), v_0(1), v_0(2), \dots) = (2, 3, 5, 9, 17, 33, \dots)$ as $v_0(m+1) = 2v_0(m) - 1$. Furthermore $v_1(m) + v_0(m) = 1$.

b) From the affine recurrence given by $A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $S^{-1}AS =$

$\text{Diag}(6, 2, 1, 0)$ with $S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ we get the exact simplex data

formula

$$SB^m S^{-1} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 3 \cdot 2^{m-1} + 3 \cdot 2^m + 2^{m-1}3^m) \\ 3(-2^{m-1} + 2^m + 2^{m-1}3^m) \\ 6^m \\ 1 \end{bmatrix}$$

which in the first coordinate gives the above sequence $3, 7, 25, \dots$ which grows like $6^m/2$.

$$(v_0(0), v_0(1), v_0(2), \dots) = (3, 7, 25, 121, 673, 3937, 23425, \dots) .$$

c) follows from the fact that in each step, a d-dimensional simplex is cut into $(d + 1)!$ simplices. This shows $v_d(G_m) = (d + 1)!v_d(G_{m-1})$. Furthermore $v_0(G_m) = \sum_j v_j(G_{m-1}) \geq v_d(G_{m-1})$. $\qquad\square$

In [46], a combinatorial interpretation of the eigenvalues $1, 2, 6$ are given. We hope it will be possible to get also exact formulas for $v_k(G_m)$ if $G_0 = K_{d+1}$ if $d \geq 3$. To do so, one could use the handshaking relation (see [24])

$$\sum_{x \in V} V_k(x) = v_{k+1}(k + 2) ,$$

where $V_k(x)$ are the number of k-dimensional complete subgraphs V_{k+1} in the unit sphere $S(x)$. In the case $k = 1$, this is the Euler handshake giving the number v_1 of edges in terms of the vertex degrees $V_0(x)$ of the graph. In the graph G_m, the spheres are all joins of smaller-dimensional spheres and in particular come in finitely many types. What one would have to do first is to give formulas for $v_k(S_{d,l,m})$ for these d-spheres of type l in level m and then get $v_k(G_m)$ by the Handshaking lemma using the fact that we know the number of these spheres. For example, in the case $d = 2$, only degrees $4, 6, 8, 12, 16, 24, ..$ can occur in the interior and degrees $3, 5, 9, 17, \dots$ at the boundary. It looks like a nice but not hopeless combinatorial challenge to analyze this.

Some numbers.
1) In the one dimensional case $G = K_2$, where $v_0(G_m) = 2^m + 1$ and $v_1(G_m) = 2^m$, we have

$$v_0(G_{m+1})/v_0(G_m) \to 2 = 2! \ .$$

2) In the two dimensional case $G = K_3$, we have everything explicit. In particular, $v_0(G_m) = 1 - 32^{m-1} + 32^m + 2^{m-1}3^m)$ which satisfies

$$v_0(G_{m+1})/v_0(G_m) \to 6 = 3! \ .$$

3) In the three dimensional case $G = K_4$, we have the following data for $\vec{v} = (v_0, v_1, v_2, v_3)$:
$\vec{v}(G) = (4, 6, 4, 1)$
$\vec{v}(G_1) = (15, 50, 60, 24)$,
$\vec{v}(G_2) = (149, 796, 1224, 576)$,
$\vec{v}(G_3) = (2745, 17000, 28080, 13824)$,
$\vec{v}(G_4) = (61649, \ldots, \ldots, 331776)$.
We don't have a closed formula for the sequence $v_0(G_m)$ starting with $4, 15, 149, 2745, \ldots$. We only know that $v_0(G_{m+1})/v_0(G_m) \to 24 = 4!$.
4) In the four dimensional case $G = K_5$, we have computed so far:
$\vec{v}(G) = (5, 10, 10, 5, 1)$,
$\vec{v}(G_1) = (31, 180, 390, 360, 120)$,
$\vec{v}(G_2) = (1081, 11340, 33300, 37440, 14400)$.
$\vec{v}(G_3) = (97561, \ldots, \ldots, \ldots, 1728000)$.
Also here, we only know $v_0(G_{m+1})/v_0(G_m) \to 120 = 5!$ and have no closed formula for the sequence $v_0(G_m)$ starting with $5, 31, 1081, 97561, \ldots$ yet.

3. Spectral distribution functions

The Laplacian of a graph is the matrix $L = B - A$, where A is the adjacency matrix and B is the diagonal degree matrix. For a complete graph K_{d+1} for example, the matrix is everywhere -1 except for the diagonal, where the entries are d. As subtracting $d+1$ produces a matrix B with d dimensional kernel and trace $-d - 1$, the eigenvalues of the Laplacian of the complete graph K_{d+1} is $d + 1$ with multiplicity d and 0 with multiplicity 1. For the house graph G for example, where a triangle K_3 (roof) is glued on top of a square C_4 leading to a graph of dimension $22/12$ as it is mixture of one and two dimensional components, the Laplacian is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix} \ .$$

While experimenting with nodal surfaces for eigenfunctions of the Laplacian on discrete 3-spheres, where we expect the nodal surface of the ground state to be a 2-sphere in general, we also computed eigenvalues of refinements and

got interested on how the eigenvalue distribution depends on refinements. While we expected some limit to be reached, we would have thought that the limit to depend on the topology of the initial graph. This is not the case. Starting with a 2-torus or a 2-sphere produces the same limiting function, as the figures illustrate and as we will show below. We can not push numerical experiments far, as the number of vertices of the refinements grows so fast. But the theory confirms this. The proof is rather elementary using a Lidskii lemma. Here is the main result:

Theorem 2 (Central limit of barycentric refinements). *The sequence $F_{G_m}(x)$ converges in $L^1([0,1])$ to a limiting distribution function $F_d(x)$ which only depends on the dimension d of the largest complete subgraph of G.*

The convergence is uniform on compact subsets in $[0, 1)$ and exponentially fast in $L^1([0, 1])$ norm or in the $L^\infty([a, b])$ norm for every compact interval $[a, b] \subset (0, 1)$. In the case $d = 1$, where we can compute the spectrum for circular graphs, we have an explicit limiting function F. We first show that the limit $F_{G_m}(x)$ exists for each graph and then prove that the limit is **universal** and only depends on the dimension d of the largest complete subgraph of G. The fact that a sequence of monotone functions f_m in $L^1([0, 1])$ which converges in $L^1([0, 1])$ converges also uniformly on every closed interval $[a, b]$ with $0 < a < b < 1$ is known in real analysis (e.g. [38]). That the convergence can not be pushed to the boundary point $x = 1$ follows from:

Lemma 3. *For $d \geq 2$, the function F_d is not in $L^\infty([0, 1])$, as the values $F_{G_m}(1)$ grow exponentially with m.*

Proof. The Courant-Fischer estimate (which is a special case of the Schur inquality in linear algebra) shows that $\lambda_{n-1} \geq \max_x(\deg(x))$, if n is the number of vertices. Therefore, for every x: then $\max(v, Lv)/(v, v) = L_{xx} = d$. Since the maximal degree increases indefinitely, the maximal eigenvalue $\lambda_{n-1} = F_{G_m}(1)$ grows, where $n = n(m)$ is the number of vertices of G_m. □

We do not know yet about the nature of the limiting functions in the case $d \geq 2$:

> **Problem:** What is the spectral nature of the limiting density of states $\mu = F'$? Does it have a discrete or singular part for $d \geq 2$?

4. THE ONE DIMENSIONAL CASE

Proposition 4. *For $d = 1$, the limiting function is $4 \sin^2(\pi x)$ so that the limiting function F is smooth.*

Proof. The eigenvalues of the Laplacian of circular graphs C_n are explicitly known using discrete Fourier transform conjugating the Laplace operator L on $l_2(Z_n)$ to the diagonal matrix with entries $2 - 2\cos(2\pi k/n) = 4\sin^2(\pi k/n)$ so that

$$\lambda_n = 4 \sin^2(\pi k/n) .$$

By definition, we have $F_{G_m}(x) \to 4\sin^2(\pi x)$. □

Proposition 5. *For $d = 1$, the Laplacian of the circular graphs satisfies the renormalization map $L(G_{m+1}) = \phi(T(L(G_m)))$, where $T(x) = 4x - x^2$ is the quadratic map with Julia set $[0,4]$ and ϕ restricts the matrix to an invariant subspace of half the dimension.*

Proof. This is a direct verification for matrices. In the Fourier picture, it becomes a double angle formula of trigonometry. Algebraically, it is an identity for Jacobi matrices, the Laplacian of the free particle on the circular graph. For $G = C_4$ for example, we have the Laplacian

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Now take the Laplacian K of C_8 and form $T(K) = 4K - K^2$. This gives

$$K = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

We see that that there are two 4-dimensional subspaces of R^8 on which K is isomorphic to the old operator L. In the above notation, we have expressed this as $\phi(K) = L$. □

Remark: The quadratic map T is associated to the Julia set of the map $x^2 - 2$ which is the "tip of the tail" parameter $z = -2 + i \cdot 0$ in the Mandelbrot set M. The polynomial T is conjugated to a Tschebychev polynomial, which by the way are the polynomials for which the Julia set is an interval [2]. Alternatively, in the context of interval maps, the map T is conjugated to the Ulam map $f(x) = 4x(1 - x)$. The density of the eigenvalues follows the arcsin probability distribution $f(x) = (x(4 - x))^{-1/2}/\pi$ supported on $[0,4]$. It has with cumulative distribution function $(2/\pi)\arcsin(\sqrt{x}/2)$. The absolutely continuous measure $\mu = f(x)1_{[0,4]}dx$ is the natural equilibrium measure on the Julia set, the measure which maximizes metric entropy and equals it to topological entropy $\log(2)$.

5. CONVERGENCE

Intuitively, universal convergence is a consequence of the fact that every simplex in G_m spans more simplices producing a self similar pattern. Smaller dimensional parts and contributions from boundary get washed away as the

refinement progresses, as in dimension d, there are exponentially more points than in dimension $d - 1$. Also in higher dimensions, we expect this to be a setup for a renormalization scheme, as the spectrum on part of the graph is close to the spectrum of the entire graph and different regions grow in the same manner.

We can not write the limiting F as a fixed point of a renormalization map yet. It can not be ruled out yet that the scalar Laplacian of G_{m+1} could be related with the form Laplacian of G_m. In two dimensions, where only 1-forms are Fermionic, the spectrum of D^2 is determined by the spectrum of the scalar 0-forms and the 2-forms.

The refinements become more and more self-similar, even so the degrees become larger and larger. We will see that the average eigenvalue $\overline{\lambda}(G) = \sum_{k=1}^{n} \lambda_k / n = ||F_G||_1$ converges for $m \to \infty$ in such a way that $\overline{\lambda}(G_{m+1}) - \overline{\lambda}(G_m)$ decreases with an exponential rate depending only on the dimension of the largest complete subgraph.

> **Problem:** Is there also in dimension $d > 1$ a functional equation for which the limiting F is a solution? Is there a relation with an equilibrium measure of a Julia set as in $d = 1$?

When we look at convergence of F_{G_m} in $L^1([0, 1])$, the limiting graph density $2v_1/v_0$ relating the number of edges v_1 with the number of vertices v_0 matters.

Lemma 6. $||F_{G_m}||_1 = \int_0^1 F_{G_m}(x) \, dx = 2v_1(G_m)/v_0(G_m)$ which is the average vertex degree of G_m.

Proof. The trace of the matrix L is the sum of the eigenvalues. By the Euler handshaking lemma, it is twice the number $v_1(G_m)$ of edges. The average eigenvalue is therefore

$$2v_1(G_m)/v_0(G_m) \ .$$

$\square$

In the case of a circular graph, we have $||F_{G_m}|| + 1 = 2$. In the case of the triangle graph, we have

$$||F_{G_m}||_1 = \frac{3 \, 2^{n+1} \, (3^n + 1)}{3 \, 2^n + 6^n + 2} \to 6 \ .$$

The fact that the average degree goes to 6 follows of also from Gauss-Bonnet, as $K(x) = 1 - d(x)/6$ is the curvature for graphs without K_4 subgraphs. Since the Euler characteristic of the triangle is 1, the sum of the curvature has to converge to 1. More generally, we have:

Corollary 7. *In any dimension d, the one has $||F_{G_m}||_1 \to (d + 1)!$ exponentially fast.*

Proof. Indeed, as the number of boundary simplices grows like $(d!)^n$ and the number of interior simplices grows like $((d+1)!)^n$, the convergence is of the order $1/(d+1)^n$. □

Since the largest vertex degree grows exponentially in m, we know $||F_{G_m}||_\infty \to \infty$. The Cheeger inequality for graphs allows to say something about the ground state energy, the smallest nonzero eigenvalue of the connected graph G.

Corollary 8. *The ground state energy $\lambda_1(G_m)$ converge to 0 exponentially fast for $m \to \infty$.*

Proof. In dimension 1, where we know the explicit spectrum and because the number of vertices grows exponentially. In general, it follows from the fact that the Cheeger number $|C(f)|/\min(|A(f)||B(f)|)$ goes to zero exponentially fast. □

In order to compare eigenvalues, one could use a perturbation result of Weyl which tells that for selfadjoint matrices A, B, the eigenvalues $\lambda_k(A+B)$ are sandwiched between $\lambda_k(A) + \lambda_1(B)$ and $\lambda_k(A) + \lambda_n(B)$. Since we need not only the individual eigenvalues to converge but also need a l_1-convergence, the following variant of the Lidskii's theorem which was used already in [26] is better suited:

Lemma 9 (Lidskii). *For any two selfadjoint complex $n \times n$ matrices A, B with eigenvalues $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, one has $\sum_{j=1}^n |\alpha_j - \beta_j| \leq \sum_{i,j=1}^n |A - B|_{ij}$.*

The reduction to the standard Lidksii theorem was told me by Yoram Last [37]:

Proof. Denote with $\gamma_i \in R$ the eigenvalues of the selfadjoint matrix $C := A - B$ and let U be the unitary matrix diagonalizing C so that $\text{Diag}(\gamma_1, \ldots, \gamma_n) = UCU^*$. We calculate

$$
\begin{aligned}
\sum_i |\gamma_i| &= \sum_i (-1)^{m_i} \gamma_i = \sum_{i,k,l} (-1)^{m_i} U_{ik} C_{kl} U_{il} \\
&\leq \sum_{k,l} |C_{kl}| \cdot |\sum_i (-1)^{m_i} U_{ik} U_{il}| \leq \sum_{k,l} |C_{kl}| .
\end{aligned}
$$

The claim follows now from Lidskii's inequality $\sum_j |\alpha_j - \beta_j| \leq \sum_j |\gamma_j|$ (see [45]) □

Proposition 10. *For any $d \geq 1$ and $G = K_{d+1}$, there exists a limiting eigenvalue distribution $F_d(x) = \lim_{m \to \infty} F_{G_m}(x)$.*

Proof. We proceed by induction. For $d = 1$, we have explicit eigenvalues. For $d = 2$, we make a refinement and divide it up into 6 pieces. Lidskii shows that the eigenvalues of G_{m+1} consist of 6 copies of the eigenvalues of G_m plus a correction term which comes from lower dimensional interfaces and is much
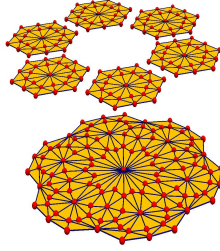
FIGURE 4.   For the proof, we cut the d-dimensional graph G_m into $(d+1)!$ smaller pieces of the form G_{m-1}. The modification on the boundary affects exponentially less vertices so that by Lidskii, the l_1 difference between the spectra goes to zero exponentially fast with m. This applies then to the $L^1([0,1])$ functions F_{G_m}.

smaller. In words, $||F_{G_{m+1,2}}|| = ||F_{G_{m,2}}|| + 6 \cdot 3||F_{G_{m,1}}||(2^n/6^n)$. As $||F_{G_{m,1}}||$ converges, the sequence $||F_{G_{m,2}}||$ is a Cauchy sequence. Now, lets go to the case $d = 3$. We have $||F_{G_{m+1,3}}|| = ||F_{G_{m,3}}|| + 24 \cdot 3||F_{G_{m,2}}||(6^n/6^{n^2})$ and again have a Cauchy sequence showing that the limit exists in $L^1([0,1])$. In each dimension we get a limiting function as the lower dimensional interfaces between the similar chambers grow exponentially slower than the chambers themselves. $\square$

Corollary 11. *There exists a limiting density of states $\mu_d = F'_d(x)$ which is a measure on $[0, 1]$.*

Proof. As F_{G_m} are monotone, it defines a measure μ_m. Since the limiting eigenvalue distribution is monotone, it defines a measure μ. $\square$

With pointwise convergence, we would have weak convergence of μ_m to μ. But we don't know that yet.

Lets now look at the proof of the theorem: we have to show that the limiting function F does not depend on the initial graph. Having seen convergence for a simplex K_{d+1}, it follows for a finite union of k simplices glued along lower dimensional simplices. Each simplex evolves in the same way under the barycentric evolution and the average is the same function F. When cutting the graph apart or disregarding lower dimensional parts, the modifications lead to change on a set of vertices which becomes exponentially less relevant as m grows as it is lower dimensional. This is justified by Lidskii's estimate as we can estimate the sum of the eigenvalue differences.

6. Vertex degree distribution

Lets look at the vertex degree distribution $d_1 \leq d_2 \leq d_3 \ldots$. In the same way as for the spectrum, we can define a degree distribution function

$$H_{G_m}(x) = d_{[xn]} .$$

The degree and eigenvalue distributions are linked by the Schur's inequality so that one can deduce the following from the eigenvalues. It is also possible to do it directly:

Corollary 12 (Limiting vertex distribution). *There exists a limiting vertex distribution function* $H(x) = \lim_{m \to \infty} H_{G_m}(x)$. *It is independent of the initial graph* G *and depends only on the dimension* d *of the largest complete subgraph of* G. *The distribution satisfies* $H(x) \geq F(x)$ *for* $x \in [0,1]$, $H(0) = F(0), H(1) = F(1)$.

Proof. Splitting the graph G_m into $(d+1)!$ smaller graphs G_{m-1} and noting that the degrees of the lower dimensional walls can be neglected in the limit as the number of vertices in those parts grows slower, we have a Cauchy sequence in L^1.
An inequality of Schur For any selfadjoint matrix A, the Schur inequality

$$\sum_{i=1}^{t} a_i \leq \sum_{i=1}^{t} \mu_i$$

holds, where μ_i are the eigenvalues of A and a_i are the diagonal elements $a_i \geq a_{i+1}$ of A. In the case $t = n$ we have equality since $\text{tr}(L) = \sum_i \lambda_i$. The Schur inequality gives $\sum_i^t d_i \geq \sum_{i=1}^t \lambda_i$ if the degrees d_i of the graph and eigenvalues λ_i of the graph are ordered in an ascending way.   □

Remark. A bit stronger than Schur is a result of Gone [1]) which assures that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \mu_i, 1 \leq k \leq n-1$$

if $\mu_1 \geq \mu_2 \geq \cdots \geq v_n = 0$ are the eigenvalues of the Laplacian of G and $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ are the vertex degrees of the graph G. This implies that $H_{G_m}(x) > F_{G_m}(x)$ for each m and suggest that also in the limit $H(x) > F(x)$ for every $x \in (0,1)$.

In the case $d = 2$, the degree d_i is related to the curvature $K(x) = 1 - d(x)/6$ which adds up to the Euler characteristic of G. We see that the curvature distribution is related to the eigenvalue distribution.

In general we can look at the curvature

$$K(x) = 1 - \frac{V_0(x)}{2} + \frac{V_1(x)}{3} - \frac{V_2(x)}{4} \cdots$$
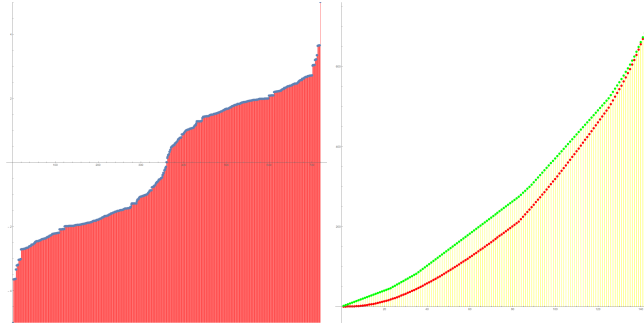
FIGURE 5. To the left is a graph of the spectral density function of the Dirac operator $d + d^*$ of a barycentric refinement of the house graph. To the right an illustration of the Schur inequality: the integrated vertex degree function is an upper bound for the integrated eigenvalue function $\int_0^x F(t)\ dt$ and agrees at the end points $0, 1$.

of a vertex which by Gauss-Bonnet-Chern [24] adds up to the Euler characteristic $\chi(G)$ of G.

The limiting distribution works also for the Dirac operator $D = d + d^*$ and so for the Hodge Laplacian $D^2 = dd^* + d^*d$. The argument is the same.

Corollary 13. *For any k, the Laplacian $L_k(G_m)$ on k-forms has a spectral limiting function which only depends on k and the dimension d of the largest simplex in $G = G_0$.*

In each dimension d, the graph G_m belonging to a simplex $G_0 = K_d$ can be cut (using some modifications on smaller dimensional sub graphs) into $(d+1)!$ smaller isomorphic pieces, which are of the form G_{m-1}. The $(d-1)$-dimensional cuts do not matter in the limit as the number of simplices in them grows exponentially less fast.

Since the matrices $D^2(G_m)$ and $L(G_{m+1})$ have the same size, we compared the distribution of the Hodge Laplacian on the level m with the distribution of the scalar Laplacian on the level $m + 1$ and they appear comparable. Because of super symmetry, the 1-form Laplacian L_1 has the same spectrum than the union of the 0 and 2-form Laplacians. The 2-form Laplacian is related to the Laplacian of the dual graph.

There might be more relations if things are extended to Schrödinger operators. In [22], we looked at renormalization maps $L \rightarrow D$ satisfying $L = D^2 + c$, where D is a Laplacian on a barycentric refined circular graph.

Its not clear whether there are algebraic relations for $d \geq 2$. For a trian-
gle $G = G_0$ with Laplacian $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ the Laplacian $L(G_1)$ of the
refinement G_1 and the form Laplacian $D^2(G_0)$ have the same size

$$L(G_1) = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix} ,$$

$$D^2(G_0) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} .$$
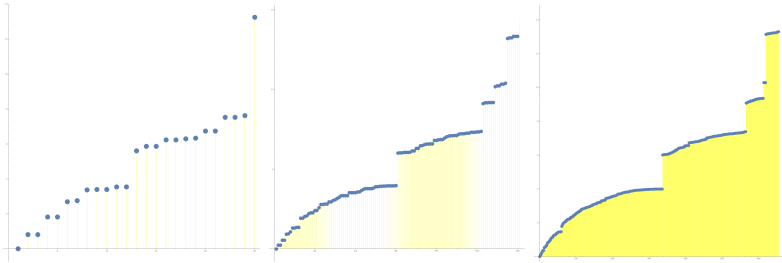
7. FIGURES

FIGURE 6.   The spectrum of the barycentric refinements
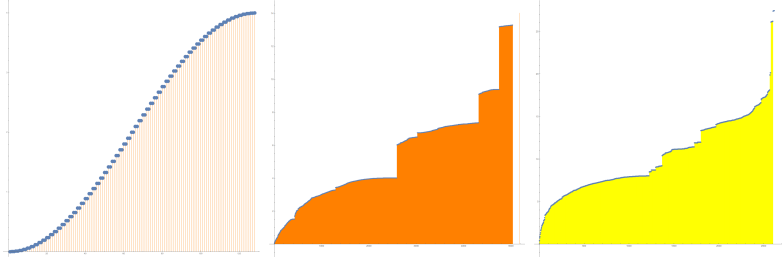G_m of the triangle, for $m = 3, 4, 5$.

FIGURE 7. The spectral functions F in the case $d = 1, d = 2$ and $d = 3$. We see the spectrum of barycentric refinements G_7 for $G = C_4$, the spectrum of the refinement G_4 of the octahedron G and finally the spectrum the refinement G_3 of the tetrahedron K_4.
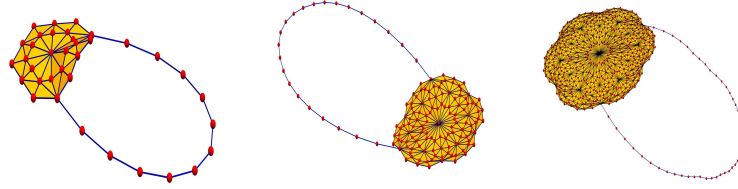

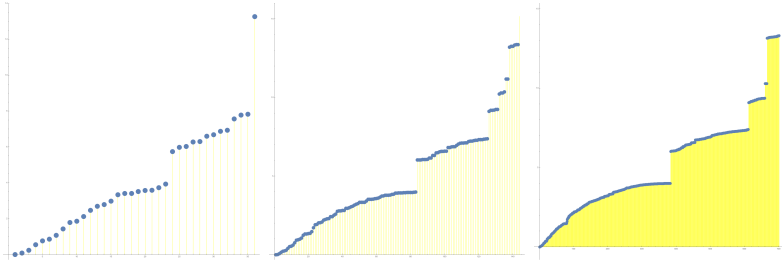FIGURE 8. Barycentric refinements of the house graph. The one dimensional component can be neglected in the limit.

FIGURE 9. The spectrum of barycentric refinements G_m of the house graphs for $m = 3, 4, 5$, which is the case $d = 2$.
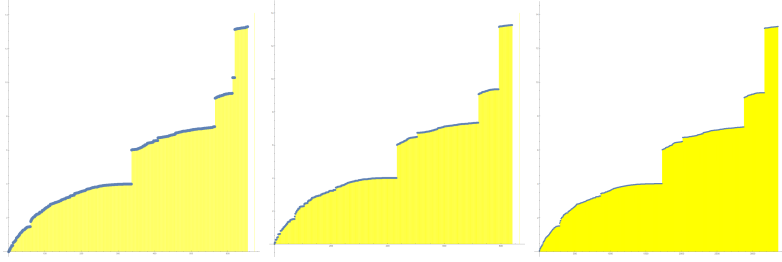
FIGURE 10.   The spectrum of barycentric refinements G_4 for the triangle the octahedron and the torus. The eigenvalue distribution look almost identical. These are all 2 dimensional cases.
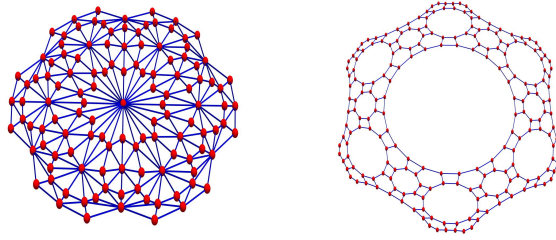

FIGURE 11.   A refinement of the triangle and its dual graph: the graph whose vertices are triangles and where two triangles are connected if they intersect in an edge. The dual graph does not have triangles.
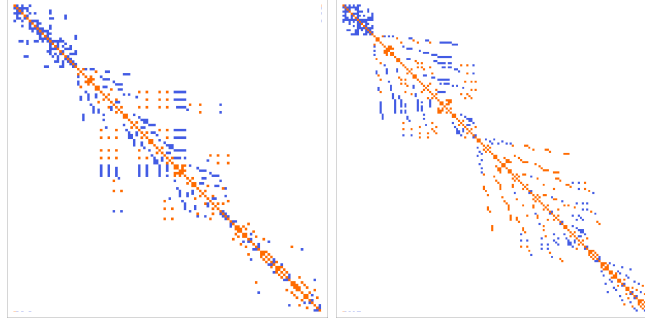

FIGURE 12.   The Hodge Laplacian $L = (d + d^*)^2$ for refinements of the triangle K_3 has three blocks L_0, L_1, L_2, where L_k acts on k-forms. The Hodge Laplacian of the tetrahedron K_4 has blocks L_0, L_1, L_2, L_3. Super-symmetry assures that the union of the spectra of the Bosonic parts L_0, L_2 is the union of the spectra of the Fermionic parts L_1, L_3.

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